Moving Points and Linearity

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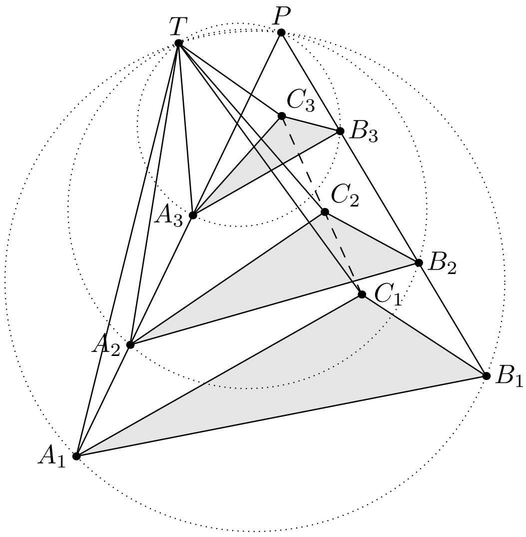
This article is about a lemma which I discovered fairly recently. I initially wanted to call it *moving points*, but unfortunately that name is already taken by another technique so I will just call it *the lemma* in this handout. This is a pretty exotic lemma, but it is nevertheless good to know and can be useful in some circumstances. Anyway here it is in all its glory.

Lemma

Let A and B be two points moving with constant velocity on two fixed lines (not necessarily distinct) ℓ_1 and ℓ_2. Let C be also a point such that the shape of $\triangle ABC$ is fixed while both A and B are moving. Then C is also moving along a fixed line with constant velocity.

Proof. This is a pretty unusual lemma, isn't it? Let t_1, t_2 and t_3 be three instances of time, and let X_i denote the position of point X at time t_i. Then it suffices to show that C_1, C_2 and C_3 are collinear. We will consider three cases as follows.

Case 1 — ℓ_1, ℓ_2 are distinct and not parallel.



First notice that

$$\frac{A_1A_2}{A_2A_3} = \frac{t_2 - t_1}{t_3 - t_2} = \frac{B_1B_2}{B_2B_3}$$

where the ratios are directed. Let T be the center of spiral similarity that sends segment A_1A_3 to segment B_1B_3. Then this also sends A_2 to B_2. Now T is also the center of spiral similarity which sends A_1B_1 to A_2B_2, and this sends C_1 to C_2 as $\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2$. Therefore,

$$\measuredangle TC_1C_2 = \measuredangle TA_1A_2.$$

Similarly, we have $\measuredangle TC_1C_3 = \measuredangle TA_1A_3 = \measuredangle TA_1A_2$, so C_1, C_2 and C_3 are collinear.

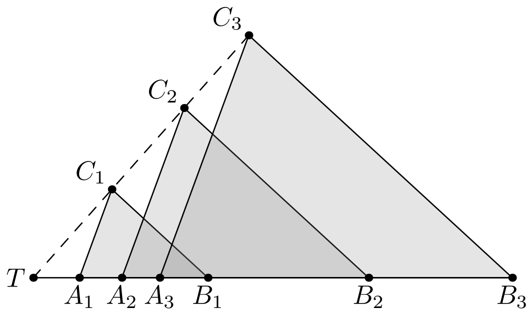
Case 2a — ℓ_1, ℓ_2 are distinct, parallel, and A and B have different velocities.

This case is the same as Case 1.

Case 2b — ℓ_1, ℓ_2 are distinct, parallel, and A and B have the same velocity.

This case can be seen as translating $\triangle ABC$ in the direction of ℓ_1 and ℓ_2 with constant velocity. Thus C obviously moves along a fixed line too.

Case 3a — $\ell_1 \equiv \ell_2$, and A and B have different velocities.



Let T be the center of homothety which maps segment A_1B_1 to segment A_2B_2. This obviously sends C_1 to C_2, so T, C_1, C_2 are collinear. As in Case 1,

$$\frac{A_1A_2}{A_2A_3} = \frac{B_1B_2}{B_2B_3}$$

so T is the center of another homothety which sends A_1B_1 to A_3B_3. Therefore, T, C_1, C_2 are collinear and the result follows.

Case 3b — $\ell_1 \equiv \ell_2$, and A and B have the same velocity.

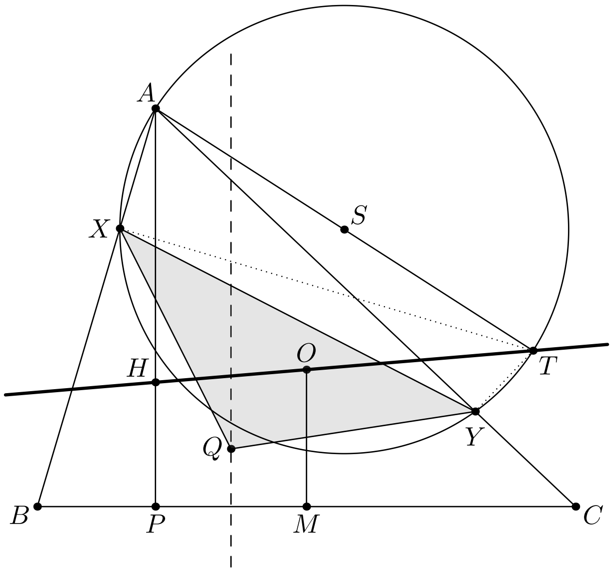
This case can be seen as moving segment AB along the line with constant velocity. Thus, obviously, C moves along a line too.

In all cases, it is easy to check that $C_1C_2/C_2C_3 = (t_2 - t_1)/(t_3 - t_2)$, so C indeed moves with constant velocity. Also, in Case 1, Case 2a and Case 3a, T can coincide with C. In that case C would be fixed, but this can be also seen as moving along a fixed line with velocity $\vec{0}$. Thus we are finally done. $\square$

Remark. In hindsight, this would have been an easy proof if I had just complex bashed instead.

Now let's destroy some problems with our lemma. The first problem is a G5 from IMO Shortlist 2016.

Problem 1. (IMO SL 2016 G5)
Let D be the foot of perpendicular from A to the Euler line (the line passing through the circumcentre and the orthocentre) of an acute scalene triangle ABC. A circle ω with centre S passes through A and D, and it intersects sides AB and AC at X and Y respectively. Let P be the foot of altitude from A to BC, and let M be the midpoint of BC. Prove that the circumcentre of triangle XSY is equidistant from P and M.



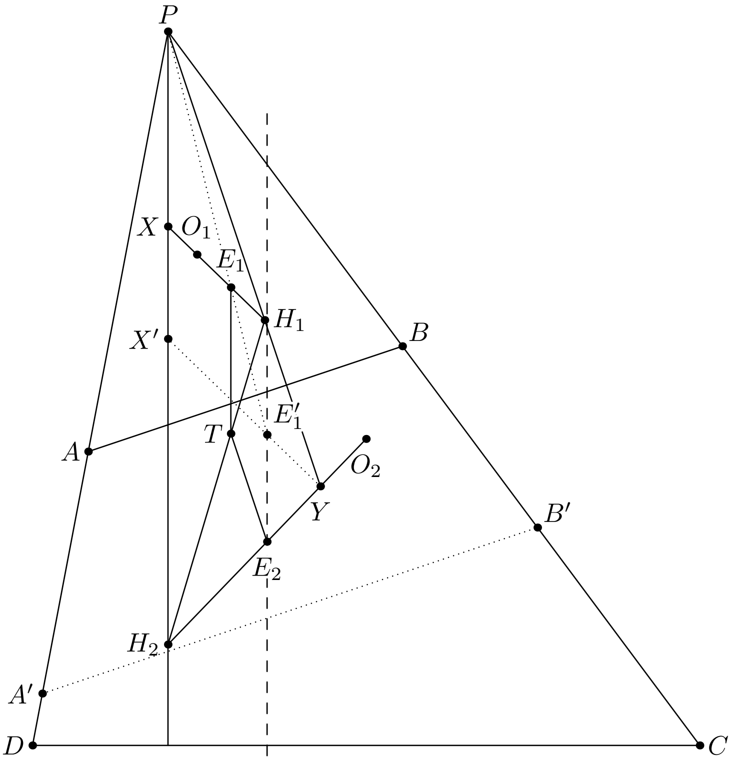
Solution. Let ω intersect OH again at T, and let's move T uniformly along the Euler line. Then both X and Y are the foots of perpendiculars from T, which means that they also move uniformly respectively. Moreover, if we let Q be the circumcenter of $\triangle XSY$, the shape of $\triangle XQY$ is fixed. This is because $\measuredangle QXY = 90^\circ - \measuredangle YSX = 90^\circ - 2\measuredangle CAB$ and similarly, $\measuredangle QYX = 90^\circ - 2\measuredangle CBA$. Therefore, by our lemma, Q moves uniformly along a fixed line too. We will show that this line is the perpendicular bisector of PM. It suffices to check for two cases that this is true. Notice that O and H are different as $\triangle ABC$ is scalene.

We will first check the case when $T = H$. Then ω is the nine-point circle of $\triangle ABC$, so Q lies on the perpendicular bisector of PM.

Now let's check the case when $T = O$. Then X and Y are midpoints of AB and AC respectively. Since $XYMP$ is an isosceles trapezoid, the perpendicular bisectors of XY and PM coincide, and hence Q lies on the perpendicular bisector of PM in this case as well. This completes the proof. □

The following problem is a bit harder but still not that bad for a G6.

Problem 2. (IMO SL 2009 G6)
Let the sides AD and BC of the quadrilateral $ABCD$ (such that AB is not parallel to CD) intersect at point P. Points O_1 and O_2 are circumcenters and points H_1 and H_2 are orthocenters of $\triangle ABP$ and $\triangle CDP$, respectively. Denote the midpoints of segments O_1H_1 and O_2H_2 by E_1 and E_2 respectively. Prove that the perpendicular from E_1 on CD, the perpendicular from E_2 on AB and the lines H_1H_2 are concurrent.



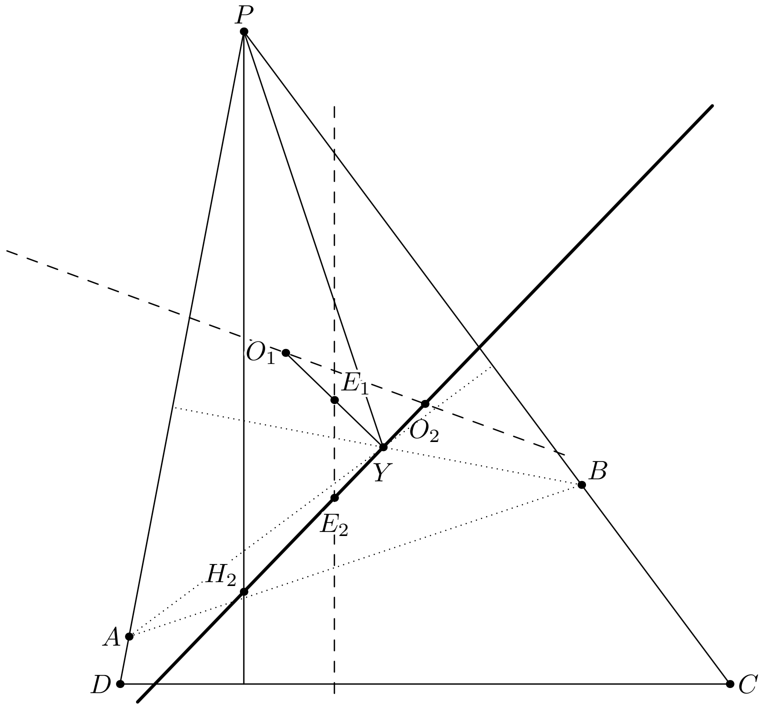
Solution. Let l_1 be the line through E_1 perpendicular to CD, and define l_2 similarly for E_2. We will first consider the case when $O_1 \neq H_1$ and $O_2 \neq H_2$. Let O_1H_1 cut the line through P perpendicular to CD at X, and define Y analogously for $\triangle PDC$. Since PH_2 and PH_1 are perpendicular to CD and AB, they are parallel to l_1 and l_2 respectively, and thus to show the concurrency, it suffices to show that

$$\frac{H_1E_1}{E_1X_1} = \frac{YE_2}{E_2H_2}.$$

Now consider a homothety centered at P which sends H_1 to Y, and denote the images of other points with $\bullet'$. Since homotheties preserve ratios, we have $H_1E_1/E_1X = YE_1'/E_1'X'$, or in other words, we must show that $E_1'E_2$ is parallel to PH_2. We can now rephrase the problem like so.

Rephrased. Let $\triangle PDC$ be a triangle with circumcenter O_2, orthocenter H_2 and nine point center E_2. Let Y be a point on $\overline{O_2H_2}$, and let A and B be points on PD and PC such that Y is the orthocenter of $\triangle PAB$. If O_1 and E_1 are the circumcenter and nine point center of that triangle, show that E_1E_2 is parallel to PH_2.

Let's move Y uniformly on $\overline{O_2H_2}$. Both A and B move uniformly since YA remains perpendicular to PC and similarly for YB. Moreover, $\triangle O_1AB$ has fixed shape since $\measuredangle APB$ is fixed, so applying the lemma shows that O_1 moves uniformly along a fixed line. This means that E_1 which is the midpoint of O_1Y also moves uniformly along a fixed line. Thus it suffices to check that this line is the line through E_2 parallel to PH_2. When $Y = H_2$, $O_1 = O_2$, which implies that $E_1 = E_2$. When $Y = O_2$, O_1 lies on PH_2 and therefore, E_1E_2 is parallel to PH_2 and we are done for this case.



Now WLOG, assume that $O_2 = H_2$. ($O_1 = H_1$ and $O_2 = H_2$ can't happen at the same time because that would imply that $AB \parallel CD$.) In that case, $\triangle ABP$ is equilateral, and hence we have $PO_1 = PH_1$ (why?), and so O_1H_1 is actually parallel to CD. Then the line through E_1 perpendicular to CD passes through $O_2 = H_2$ and hence the three lines are concurrent at this point. We are finally done. $\square$