Some Properties of the Feuerbach Point

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In this note, we provide a completely synthetic proof to the following theorem and some of its consequences.

Theorem 1

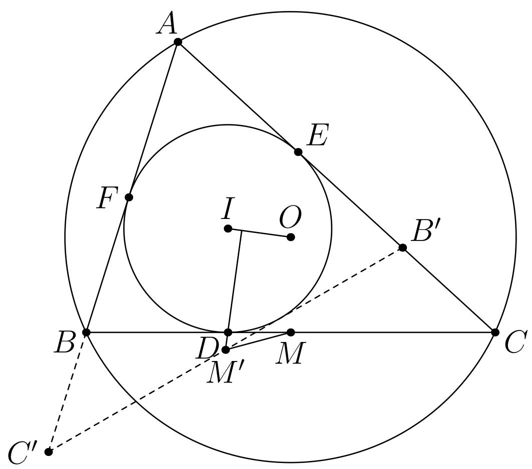
The incircle and the nine-point circle of a triangle are internally tangent to each other, and this tangency point is known as the *Feuerbach point*.

Similar properties hold for the three excircles too. There are numerous ways to show this, the shortest using Casey's theorem, but here we prove it completely synthetically. To do so, we need a neat lemma beforehand.

Lemma 2

Let $\triangle ABC$ be a triangle with circumcenter O, incenter I and let the incircle touch side BC at D. Let M be the midpoint of BC and let M' be the reflection of M over $\overline{AI}$. Then $M'D \perp OI$.

We will present two ways to prove this lemma. The first one is a straightforward proof using linearity of power of a point. The second one is purely synthetic but requires more observations.



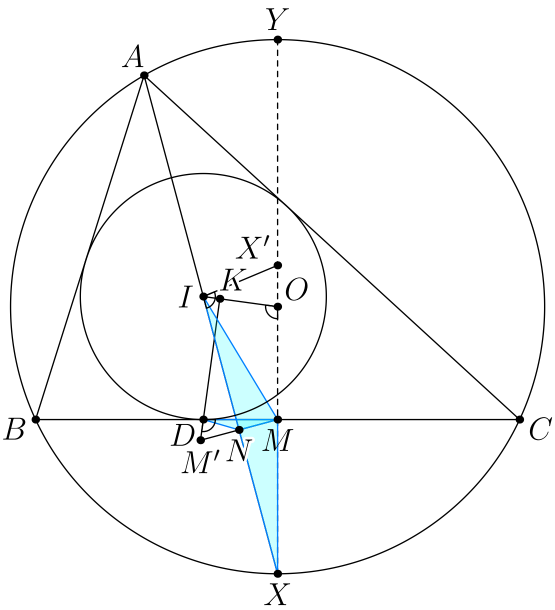
First approach using linearity. Without loss of generality, assume that $AC > AB$. Let ω and Γ be the incircle and circumcircle of $\triangle ABC$ respectively. Let B' and C' be the reflections of B and C over $\overline{AI}$, and let the incircle touch sides AC and AB at E and F. Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(X) = \mathrm{pow}(X, \Gamma) - \mathrm{pow}(X, \omega),$$

where X is any point in the plane. It is easy to show that f is linear. Therefore,

$$\begin{aligned} 2f(M') &= f(B') + f(C') \\ &= -B'A \cdot B'C - B'E^2 + C'B \cdot C'A - C'F^2 \\ &= (AC - AB)^2 - BF^2 - CE^2 \\ &= (CE - BF)^2 - BF^2 - CE^2 \\ &= -2BF \cdot CE \\ &= -2BD \cdot DC \\ &= 2f(D) \end{aligned}$$

and hence it follows that $f(M') = f(D) \implies M'O^2 - M'I^2 = DO^2 - DI^2$ and so $\overline{M'D}$ is perpendicular to $\overline{OI}$. $\square$



Second approach using similar triangles. Let X, Y, and N be the midpoints of arc BC, arc BAC and segment MM' respectively. Let $\overline{M'D}$ meet $\overline{OI}$ at K. Since $\angle IDM = \angle INM = 90^\circ$, $IDNM$ is cyclic so $\angle MIX = \angle NDM$. Moreover, since $\angle DMX = 90^\circ$, $\angle IXM = \angle DMN$, so $\triangle MIX \sim \triangle NDM$. Let X' be the reflection of X over M. Then since M' is the reflection of M over N, it follows that $\triangle X'IX$ and $\triangle M'DM$ are also similar. Now by the incenter lemma,

$$XI^2 = XB^2 = XM \cdot XY = 2XM \cdot \frac{XY}{2} = XX' \cdot XO$$

and hence $\triangle XIX' \sim \triangle XOI$. Therefore,

$$\angle M'DM = \angle X'IX = \angle IOX = \angle KOM,$$

so $KOMD$ is cyclic. This implies that $\angle DKO = 90^\circ$. $\square$

We will also need this fact about isogonal conjugation.

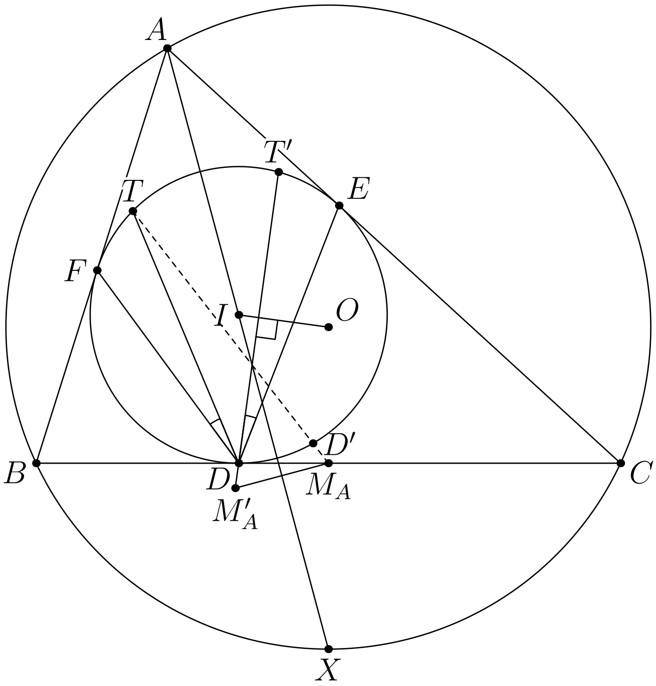
Lemma 3

The isogonal conjugate of a point with respect to $\triangle ABC$ lies on (ABC) if and only if it is a point at infinity.

The proof is really easy, so we leave this as an exercise to the reader.

Now let's take an attempt on showing the actual tangency. As usual with these types of problems, we will try to find two triangles $\mathcal{P}_1$ and $\mathcal{P}_2$, lying on the incircle and nine-point circle respectively, such that they are homothetic and their homothetic center lies on one of the circles. (Convince yourself why this implies the tangency.)

Proof of Theorem 1. Let M_A, M_B and M_C be the midpoints of BC, CA and AB respectively, and let D', E' and F' be the symmetric points of D, E, F with respect to $\overline{AI}$, $\overline{BI}$ and $\overline{CI}$ respectively. We claim that $\triangle D'E'F'$ is the desired homothetic triangle. To see the homothety, just notice that $DE' = EF = DF'$, so $E'F' \parallel BC \parallel M_BM_C$. It remains to show that the homothetic center lies on the incircle.



In fact, we claim that this homothetic center is the isogonal conjugate with respect to $\triangle DEF$ of the point at infinity perpendicular to $\overline{OI}$, say T. By lemma 3, this point lies on the incircle. Let T' be the symmetric point of T with respect to $\overline{AI}$. Let M'_A be the reflection of M_A over $\overline{AI}$. Then as $\overline{DT}$ and $\overline{DT'}$ are isogonal in $\angle D$, by definition of T it follows that $DT' \perp OI$. Therefore, T', D and M'_A are collinear by lemma 2, and reflecting over $\overline{AI}$ shows that T, D', M_A are collinear. Similarly, T also lies on $\overline{M_BE'}$ and $\overline{M_CF'}$ so the proof is complete. $\square$

As stated before, the point T is called the Feuerbach point of $\triangle ABC$. This approach of showing the tangency leads to an important result without much difficulty. We recall a well-known construction before presenting it.

Lemma 4

Let P be a point on the circumcircle of $\triangle ABC$, and let P_A, P_B, P_C be the reflections of P over $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$. Then P_A, P_B and P_C all lie on a single line ℓ, and ℓ passes through the orthocenter H of $\triangle ABC$. We call ℓ the *Steiner line* of P, and P the *anti-Steiner point* of ℓ.

Proof. This theorem looks very intimidating, but it is mostly just angle chasing. Let Q be the isogonal conjugate of P with respect to $\triangle ABC$. By Lemma 3, this point is a point at infinity. We now claim that $\overline{P_A H}$ is perpendicular to the direction of Q; this immediately implies the theorem as there is only one line through H perpendicular to the direction of Q. Let $\overline{PP_A}$ intersect (ABC) at K, and let P' be the point on (ABC) such that $PP' \parallel BC$. Since $\overline{AP'}$ and $\overline{AP}$ are isogonal in $\angle A$, it follows that Q lies on $\overline{AP'}$. By the orthocenter lemma, it is also easy to see that AKP_AH is a parallelogram. Finally,

$$\angle(\overline{HP_A}, \overline{AP'}) = \angle(\overline{AK}, \overline{AP'}) = \angle(\overline{PK}, \overline{PP'}) = \angle(\overline{PK}, \overline{BC}) = 90^\circ$$

so we are done. □

Now we can state the following beautiful result.

Theorem 5

T is the anti-Steiner point of $\overline{OI}$ with respect to $\triangle DEF$.

Of course, for $\overline{OI}$ to have an anti-Steiner point, it must pass through the orthocenter of $\triangle DEF$. However, this is a well-known result, and an important exercise if you haven't seen its proof so we won't prove it here.

Proof of Theorem 5. This is almost an immediate consequence of Lemma 4. From the proof of Theorem 1, we know that the isogonal conjugate of T is the point at infinity in the direction perpendicular to line $\overline{OI}$. From Lemma 4, we know that the Steiner line of T is the line through the orthocenter of $\triangle DEF$ parallel to $\overline{OI}$. But $\overline{OI}$ passes through the orthocenter so the proof is finished. □